AD FALCON API Manual

# Theory of Orthotropic Linear Elasticity

Javad Ghorbani

March 14, 2026

# Contents

# 1 Theory of Orthotropic Linear Elasticity

This note describes the orthotropic linear elastic constitutive model implemented in FALCON.

---

## 1.1 Syntax

This model is typically provided as a user-defined mechanical material. In `% Materials`, configure it using `@UMAT:` with category `Mechanical`, passing parameters as `name=value` pairs.

Example:

```
@UMAT: path/to/orthotropic.cpp path/to/orthotropic.hpp Mechanical E1=1e8
E2=5e7 E3=5e7 Nu12=0.25 Nu23=0.25 Nu31=0.25 G12=3e7 G23=3e7 G31=3e7
```

Use the parameter names shown in the table below.

## 1.2 Material parameters

| Symbol | Keyword in input | Units | Required | Description |
|---|---|---|---|---|
| $E_1$ | `E1` | stress | ✓ | Young's modulus in the 1-direction. |
| $E_2$ | `E2` | stress | ✓ | Young's modulus in the 2-direction. |
| $E_3$ | `E3` | stress | ✓ | Young's modulus in the 3-direction. |
| $\nu_{12}$ | `Nu12` | – | ✓ | Major Poisson's ratio (strain in direction 2 due to stress in direction 1). |
| $\nu_{23}$ | `Nu23` | – | ✓ | Major Poisson's ratio (strain in direction 3 due to stress in direction 2). |

| Symbol | Keyword in input | Units | Required | Description |
|---|---|---|---|---|
| $\nu_{31}$ | Nu31 | – | ✓ | Major Poisson's ratio (strain in direction 1 due to stress in direction 3). |
| $G_{12}$ | G12 | stress | ✓ | Shear modulus in the 1-2 plane. |
| $G_{23}$ | G23 | stress | ✓ | Shear modulus in the 2-3 plane. |
| $G_{31}$ | G31 | stress | ✓ | Shear modulus in the 3-1 plane. |

---



## 1.3 Constitutive relation

Orthotropic materials exhibit elastic symmetry with respect to three mutually perpendicular planes. The constitutive relation is expressed in Voigt notation (order: $[11, 22, 33, 23, 13, 12]$) as:

$$\sigma = C\varepsilon \tag{1}$$

where $\sigma$ is the stress vector, $\varepsilon$ is the strain vector, and $C$ is the 6×6 stiffness matrix.

### 1.3.1 Compliance matrix

The compliance matrix $S = C^{-1}$ is constructed from the engineering constants as:

$$S = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & -\frac{\nu_{13}}{E_1} & 0 & 0 & 0 \\ -\frac{\nu_{21}}{E_2} & \frac{1}{E_2} & -\frac{\nu_{23}}{E_2} & 0 & 0 & 0 \\ -\frac{\nu_{31}}{E_3} & -\frac{\nu_{32}}{E_3} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{31}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \tag{2}$$

### 1.3.2 Poisson's ratio reciprocity

The compliance matrix must be symmetric to satisfy thermodynamic requirements. This enforces the following reciprocity relations:

$$\frac{\nu_{21}}{E_2} = \frac{\nu_{12}}{E_1}, \quad \frac{\nu_{32}}{E_3} = \frac{\nu_{23}}{E_2}, \quad \frac{\nu_{13}}{E_1} = \frac{\nu_{31}}{E_3} \tag{3}$$

which can be expressed as:

$$\nu_{21} = \nu_{12}\frac{E_2}{E_1}, \quad \nu_{32} = \nu_{23}\frac{E_3}{E_2}, \quad \nu_{13} = \nu_{31}\frac{E_1}{E_3} \tag{4}$$

These relations are automatically enforced in the FALCON implementation. The user provides the three major Poisson's ratios ($\nu_{12}, \nu_{23}, \nu_{31}$), and the minor ratios are computed internally.

### 1.3.3 Stiffness matrix

The stiffness matrix $\mathbf{C}$ is obtained by inverting the compliance matrix:

$$\mathbf{C} = \mathbf{S}^{-1} \tag{5}$$

The inversion is performed using Gauss–Jordan elimination with partial pivoting to ensure numerical stability. The resulting stiffness matrix is stored for efficient stress computation during incremental loading.

> **Note**: The compliance matrix must be positive definite (i.e., all eigenvalues positive) for the material to be physically admissible. This requires that the Poisson's ratios satisfy certain constraints. In FALCON, basic sanity checks are performed to ensure rough admissibility ($-0.99 < \nu_{ij} < 0.49$), but the user should verify that the material parameters yield a positive definite compliance matrix.

---

## 1.4 Stress–strain relation

The incremental stress–strain relation is:

$$\Delta\boldsymbol{\sigma} = \mathbf{C}\Delta\boldsymbol{\varepsilon} \tag{6}$$

where $\Delta\boldsymbol{\sigma}$ is the stress increment and $\Delta\boldsymbol{\varepsilon}$ is the strain increment. The implementation uses Voigt notation with the ordering $[11, 22, 33, 23, 13, 12]$.

The stiffness matrix $\mathbf{C}$ has the general structure:

$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \tag{7}$$

where the normal–normal coupling terms $C_{ij}$ (for $i, j \in \{1, 2, 3\}$) are obtained from the inverted compliance matrix. Since the compliance matrix has a block-diagonal structure with uncoupled shear components, the shear terms in the stiffness matrix are: - $C_{44} = G_{23}$ - $C_{55} = G_{31}$ - $C_{66} = G_{12}$

---

## 1.5 Parameter constraints

For the material to be physically admissible, the compliance matrix must be **positive definite**. This requires:

1. **Positive moduli**: All Young's moduli and shear moduli must be positive:

$$E_1 > 0, \quad E_2 > 0, \quad E_3 > 0, \quad G_{12} > 0, \quad G_{23} > 0, \quad G_{31} > 0$$

2. **Poisson's ratio bounds**: The Poisson's ratios must satisfy constraints that ensure positive definiteness. A necessary (but not sufficient) condition is:

$$-1 < \nu_{ij} < 0.5$$

More stringent constraints depend on the specific values of $E_1, E_2, E_3$ and include:

$$\nu_{12}^2 < \frac{E_1}{E_2}, \quad \nu_{23}^2 < \frac{E_2}{E_3}, \quad \nu_{31}^2 < \frac{E_3}{E_1}$$

and the determinant condition:

$$1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - 2\nu_{12}\nu_{23}\nu_{31} > 0$$

FALCON performs basic sanity checks on the input parameters, but the user should verify that the material parameters yield a positive definite compliance matrix. If the compliance matrix is singular or not positive definite, the inversion will fail and an error will be raised.

---

## 1.6 Special cases

### 1.6.1 Transversely isotropic material

A **transversely isotropic** material has rotational symmetry about one axis (the 1-direction in this case). The material parameters satisfy:

- $E_2 = E_3 = E_t$ (transverse Young's modulus)
- $E_1 = E_l$ (longitudinal Young's modulus)
- $\nu_{12} = \nu_{31} = \nu_{lt}$ (major Poisson's ratio)
- $\nu_{23} = \nu_t$ (transverse Poisson's ratio)
- $G_{12} = G_{31} = G_{lt}$ (longitudinal-transverse shear modulus)
- $G_{23} = G_t = E_t/[2(1 + \nu_t)]$ (transverse shear modulus)

The compliance matrix reduces to:

$$\mathbf{S} = \begin{bmatrix} \frac{1}{E_l} & -\frac{\nu_{lt}}{E_l} & -\frac{\nu_{lt}}{E_l} & 0 & 0 & 0 \\ -\frac{\nu_{lt}}{E_l} & \frac{1}{E_t} & -\frac{\nu_t}{E_t} & 0 & 0 & 0 \\ -\frac{\nu_{lt}}{E_l} & -\frac{\nu_t}{E_t} & \frac{1}{E_t} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_t} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{lt}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{lt}} \end{bmatrix} \tag{8}$$

The corresponding stiffness matrix $\mathbf{C} = \mathbf{S}^{-1}$ has the structure:

$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{12} & C_{23} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{55} \end{bmatrix} \tag{9}$$

where the non-zero components are: - $C_{11} = \frac{E_l(1-\nu_t)}{(1+\nu_t)(1-\nu_t-2\nu_{lt}^2 E_t/E_l)}$ - $C_{22} = \frac{E_t(1-\nu_{lt}^2 E_t/E_l)}{(1+\nu_t)(1-\nu_t-2\nu_{lt}^2 E_t/E_l)}$ - $C_{12} = \frac{E_t \nu_{lt}}{1-\nu_t-2\nu_{lt}^2 E_t/E_l}$ - $C_{23} = \frac{E_t(\nu_t+\nu_{lt}^2 E_t/E_l)}{(1+\nu_t)(1-\nu_t-2\nu_{lt}^2 E_t/E_l)}$ - $C_{44} = G_t = \frac{E_t}{2(1+\nu_t)}$ - $C_{55} = G_{lt}$

### 1.6.2 Isotropic material

An **isotropic** material has identical properties in all directions. The material parameters satisfy: - $E_1 = E_2 = E_3 = E$ (Young's modulus) - $\nu_{12} = \nu_{23} = \nu_{31} = \nu$ (Poisson's ratio) - $G_{12} = G_{23} = G_{31} = G = \frac{E}{2(1+\nu)}$ (shear modulus)

The compliance matrix becomes:

$$\mathbf{S} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix} \tag{10}$$

The corresponding stiffness matrix is:

$$\mathbf{C} = \begin{bmatrix} K+\frac{4}{3}G & K-\frac{2}{3}G & K-\frac{2}{3}G & 0 & 0 & 0 \\ K-\frac{2}{3}G & K+\frac{4}{3}G & K-\frac{2}{3}G & 0 & 0 & 0 \\ K-\frac{2}{3}G & K-\frac{2}{3}G & K+\frac{4}{3}G & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix} \tag{11}$$

where $K = \frac{E}{3(1-2\nu)}$ is the bulk modulus and $G = \frac{E}{2(1+\nu)}$ is the shear modulus. This can also be expressed in terms of the Lamé parameters $\lambda$ and $\mu$:

$$\mathbf{C} = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \tag{12}$$

where $\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$ and $\mu = G$.

---

## 1.7 References (selection)


- Lekhnitskii, S. G. (1981). *Theory of Elasticity of an Anisotropic Body*. Mir Publishers.
- Ting, T. C. T. (1996). *Anisotropic Elasticity: Theory and Applications*. Oxford University Press.