AD FALCON API Manual

Fully Coupled Dynamic Analysis with FALCON

Javad Ghorbani

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1 Fully Coupled Dynamic Analysis with FALCON

1.1 Introduction

This reference summarises the governing equations for unsaturated-soil dynamics—mass, momentum, and energy balances coupled with Darcy flow and Bishop's effective stress. All symbols used below are listed once in **Table 2**.

1.2 Conservation of Mass (see Schrefler and Scotta (2001))

For each phase $\alpha \in \{s, w, g\}$

$$\frac{\partial(\rho_\alpha n^\alpha)}{\partial t} + \nabla \cdot \left(\rho_\alpha n^\alpha \dot{\mathbf{u}}_\alpha\right) = 0 \tag{1}$$

With $\dot{\mathbf{u}}_s = \dot{\mathbf{u}}$ and Darcy velocity $\dot{\mathbf{w}}^\beta$

$$\frac{Dn}{Dt} = (1 - n)\nabla \cdot \dot{\mathbf{u}} \tag{2}$$

Fluid phases $\beta \in \{w, g\}$

$$\frac{nS_\beta}{K_\beta}\frac{Dp_\beta}{Dt} + n\frac{DS_\beta}{Dt} + S_\beta \nabla \cdot \dot{\mathbf{u}} + \nabla \cdot \dot{\mathbf{w}}^\beta + \dot{\mathbf{w}}^\beta \cdot \frac{\nabla \rho_\beta}{\rho_\beta} = 0 \tag{3}$$

1.3 Linear Momentum Balance

$$\nabla \cdot \boldsymbol{\sigma} + \rho\,\mathbf{b} = \rho\,\ddot{\mathbf{u}} + \rho^g\,\ddot{\mathbf{u}}_{ws} + \rho^w\,\ddot{\mathbf{u}}_{gs} \tag{4}$$

1.4 Balance of Energy

$$\frac{DE_I}{Dt} = \mathcal{P} - \frac{DE_k}{Dt} \tag{5}$$

$$\frac{DE_k}{Dt} = \sum_\alpha \int_\Omega \rho^\alpha \ddot{\mathbf{u}}_\alpha \cdot \dot{\mathbf{u}}_\alpha \, d\Omega \tag{6}$$

$$\mathcal{P} = \sum_\alpha \left[\int_\Gamma \mathbf{t}^\alpha \cdot \dot{\mathbf{u}}_\alpha \, d\Gamma + \int_\Omega (\rho^\alpha \mathbf{b} + \mathbf{I}^\alpha) \cdot \dot{\mathbf{u}}_\alpha \, d\Omega\right] \tag{7}$$

1.5 Effective Stress

Unsaturated effective stress is evaluated with an α**-enriched Bishop formulation** (see Ghorbani and Kodikara (2024)):

$$\boldsymbol{\sigma}' = \boldsymbol{\sigma} + \chi\, p_w\, \mathbf{I} + (1-\chi)\, p_g\, \mathbf{I} \tag{8}$$

with $\chi = \alpha_{\text{eff}}\, S_w$

Table 1: Bishop Effective Stress Formulations

Variant	Expression	Notes
α**-enriched (FALCON default)**	$\alpha_{\text{eff}}(S_w) = S_w^{\frac{\beta_1}{S_w^{\beta_2}}-1}, \quad \chi = S_w^{\frac{\beta_1}{S_w^{\beta_2}}}$	parameters β_1, β_2 tune factors such as the decline of capillary contribution toward dryness

Key properties
* $\chi \to 1$ at saturation, $\chi \to 0$ at full dryness.
* Choosing $\beta_1 = 1$ and $\beta_2 = 0$ yields $\chi = S_w$ (classical Bishop).
* The relationship between χ and S_w is assumed independent of hydraulic hysteresis.

These compact relations replace the lengthy derivations while retaining the essential variables for constitutive modelling.

1.6 Work-Conjugate Pairs and FEM Formulation

1.6.1 Internal-Energy Rate

$$\frac{De_I}{Dt} = \boldsymbol{\sigma}' : \nabla\dot{\mathbf{u}} + \sum_{\beta=w,g} n\alpha_{\text{eff}}^{\beta} p_\beta \dot{S}_\beta + R \tag{9}$$

$$R = \sum_{\beta=w,g} p_\beta \left[\frac{n\alpha_{\text{eff}}^{\beta} S_\beta}{K_\beta} \frac{Dp_\beta}{Dt} + \alpha_{\text{eff}}^{\beta} \frac{\dot{\mathbf{w}}^\beta}{\rho^\beta} \cdot \nabla\rho^\beta \right] \tag{10}$$

Neglecting compressibility and convective terms

$$\frac{De_I}{Dt} = \boldsymbol{\sigma}' : \nabla\dot{\mathbf{u}} - n\alpha_{\text{eff}}^{w} p_c \dot{S}_w \tag{11}$$

1.7 Weak Forms

1.7.1 Fluid mass balance

$$\int_{\Omega} \delta p_w A_w \, d\Omega + \int_{\Omega} \delta p_g A_g \, d\Omega + \int_{\Gamma_{qw}} \delta p_w (\dot{\mathbf{w}}^w \cdot \mathbf{n}^* - \overline{\mathbf{w}}^w) \, d\Gamma - \int_{\Gamma_{qg}} \delta p_g (\dot{\mathbf{w}}^g \cdot \mathbf{n}^* - \overline{\mathbf{w}}^g) \, d\Gamma = 0 \tag{13}$$

with

$$A_\beta = \frac{nS_\beta}{K_\beta} \frac{Dp_\beta}{Dt} + n\frac{DS_\beta}{Dt} + S_\beta \nabla \cdot \dot{\mathbf{u}} + \nabla \cdot \dot{\mathbf{w}}^\beta + \dot{\mathbf{w}}^\beta \cdot \frac{\nabla \rho_\beta}{\rho_\beta} \tag{14}$$

1.7.2 Generalized Darcy law (see Zienkiewicz et al. (1990))

$$\dot{\mathbf{w}}^\beta = \mathbf{k}_\beta [-\nabla p_\beta + \rho_\beta (\mathbf{b} - \ddot{\mathbf{u}})], \qquad \mathbf{k}_\beta = \mathbf{k}_{\text{int}} k_{r\beta} / \eta_\beta \tag{15}$$

1.7.3 Solid momentum balance

$$\int_{\Omega} \delta \mathbf{u}^\top (\mathbf{L}^\top \boldsymbol{\sigma} + \rho \mathbf{b} - \rho \ddot{\mathbf{u}}) \, d\Omega + \int_{\Gamma_t} \delta \mathbf{u}^\top (\bar{\mathbf{t}} - \mathbf{I}_\sigma^\top \boldsymbol{\sigma}) \, d\Gamma = 0 \tag{16}$$

1.8 FE Discrete Equations

Unknown vectors: $\mathbf{U}, \mathbf{P}_w, \mathbf{P}_g$.

$$\mathbf{M}_u \ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} - \mathbf{Q}_w^* \mathbf{P}_w - \mathbf{Q}_g^* \mathbf{P}_g = \mathbf{F}_u \tag{17}$$

$$\mathbf{M}_w \ddot{\mathbf{U}} + \mathbf{Q}_w^\top \dot{\mathbf{U}} + \mathbf{C}_{ww} \dot{\mathbf{P}}_w + \mathbf{C}_{wg} \dot{\mathbf{P}}_g + \mathbf{H}_{ww} \mathbf{P}_w = \mathbf{F}_w \tag{18}$$

$$\mathbf{M}_g \ddot{\mathbf{U}} + \mathbf{R}_g^\top \dot{\mathbf{U}} + \mathbf{C}_{gw} \dot{\mathbf{P}}_w + \mathbf{C}_{gg} \dot{\mathbf{P}}_g + \mathbf{H}_{gg} \mathbf{P}_g = \mathbf{F}_g \tag{19}$$

(See Appendix A for matrix definitions.)

1.8.1 Reduced System: Negligible Fluid-Phase Inertia ($\mathbf{M}_w \approx \mathbf{0},\ \mathbf{M}_g \approx \mathbf{0}$)

In many geomechanics applications the inertia of pore fluids is negligible compared with the solid skeleton. Assuming $\mathbf{M}_w \approx \mathbf{0}$ and $\mathbf{M}_g \approx \mathbf{0}$, the discrete equations reduce to

$$\mathbf{M}_u \ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} - \mathbf{Q}_w^* \mathbf{P}_w - \mathbf{Q}_g^* \mathbf{P}_g = \mathbf{F}_u \tag{17R}$$

$$\mathbf{Q}_w^\top \dot{\mathbf{U}} + \mathbf{C}_{ww} \dot{\mathbf{P}}_w + \mathbf{C}_{wg} \dot{\mathbf{P}}_g + \mathbf{H}_{ww} \mathbf{P}_w = \mathbf{F}_w \tag{18R}$$

$$\mathbf{R}_g^\top \dot{\mathbf{U}} + \mathbf{C}_{gw} \dot{\mathbf{P}}_w + \mathbf{C}_{gg} \dot{\mathbf{P}}_g + \mathbf{H}_{gg} \mathbf{P}_g = \mathbf{F}_g \tag{19R}$$

These reduced equations are used by default in FALCON unless the inclusion of fluid-phase inertia is explicitly required.

1.9 Appendix A: Finite-Element Matrices and Vectors

1.9.1 Mass matrices

$$\mathbf{M}_u = \int_\Omega \mathbf{N}_u^\top \rho\, \mathbf{N}_u \, d\Omega \tag{20}$$

$$\mathbf{M}_w = \int_\Omega (\nabla \mathbf{N}_{pw})^\top \left(\mathbf{k}_g \rho_g + \mathbf{k}_w \rho_w\right) \mathbf{N}_u \, d\Omega \tag{21}$$

$$\mathbf{M}_g = \int_\Omega (\nabla \mathbf{N}_{pg})^\top \mathbf{k}_g \rho_g \, \mathbf{N}_u \, d\Omega \tag{22}$$

1.9.2 Coupling matrices

At the element level, the coupling matrices that appear in Eqs. (17)–(19) are assembled from Gauss-point contributions and can be written in compact form as

$$\mathbf{Q}_w^* = \int_\Omega \mathbf{B}^\top \left(\chi\, \mathbf{m} - \mathbf{S}_{ep}\right) \mathbf{N}_{pw} \, d\Omega, \quad \mathbf{Q}_g^* = \int_\Omega \mathbf{B}^\top \left((1-\chi)\, \mathbf{m} + \mathbf{S}_{ep}\right) \mathbf{N}_{pg} \, d\Omega \tag{23}$$

for the stress-pressure couplings in the solid momentum balance, and

$$\mathbf{Q}_w = \int_\Omega \left(S_w + e\, \tfrac{\partial S_w}{\partial e}\right) \mathbf{B}^\top \mathbf{m}\, \mathbf{N}_{pw} \, d\Omega, \quad \mathbf{R}_g = \int_\Omega \left(1 - S_w - e\, \tfrac{\partial S_w}{\partial e}\right) \mathbf{B}^\top \mathbf{m}\, \mathbf{N}_{pg} \, d\Omega \tag{24}$$

for the velocity–pressure couplings that enter the fluid mass balances. Here $\mathbf{m}$ is the volumetric projector (see Table 2), χ is the Bishop–type weighting factor from the effective-stress model, and $\mathbf{S}_{ep}$ is the suction-coupling vector returned by the UMAT interface (see UMAT development and the GCC model example).

In the saturated limit with negligible saturation derivatives ($S_w \to 1,\ \partial S_w/\partial e \to 0,\ \mathbf{S}_{ep} \to \mathbf{0}$), these expressions reduce to the classical Biot-type forms based on $\mathbf{B}^\top \mathbf{m}\, \mathbf{N}_{pw}$.

1.9.3 Hydraulic conductivity matrices

$$\mathbf{H}_{ww} = \int_\Omega (\nabla \mathbf{N}_{pw})^\top \mathbf{k}_w \nabla \mathbf{N}_{pw} \, d\Omega \tag{25}$$

$$\mathbf{H}_{gg} = \int_\Omega (\nabla \mathbf{N}_{pg})^\top \mathbf{k}_g \nabla \mathbf{N}_{pg} \, d\Omega \tag{26}$$

1.9.4 Compressibility matrices

$$\mathbf{C}_{ww} = \int_\Omega \mathbf{N}_{pw}^\top \mathbf{C}_1^* \mathbf{N}_{pw} \, d\Omega \tag{27}$$

$$\mathbf{C}_{gg} = \int_{\Omega} \mathbf{N}_{pg}^{\top} \mathbf{C}_2^* \mathbf{N}_{pg} \, d\Omega \tag{28}$$

$$\mathbf{C}_{wg} = \int_{\Omega} \mathbf{N}_{pw}^{\top} \mathbf{C}_3^* \mathbf{N}_{pg} \, d\Omega \tag{29}$$

$$\mathbf{C}_{gw} = \int_{\Omega} \mathbf{N}_{pg}^{\top} \mathbf{C}_3^* \mathbf{N}_{pw} \, d\Omega \tag{30}$$

$$\mathbf{R}_c = \int_{\Omega} \mathbf{B}^{\top} \mathbf{C}_6 \mathbf{m} \mathbf{N}_{pg} \, d\Omega \tag{31}$$

1.9.5 Coefficient matrices

$$\mathbf{C}_1^* = \tfrac{nS_w}{K_w} - n\frac{\partial S_w}{\partial p_c} \tag{32}$$

$$\mathbf{C}_2^* = \tfrac{nS_g}{K_g} - n\frac{\partial S_w}{\partial p_c} \tag{33}$$

$$\mathbf{C}_3^* = n\frac{\partial S_w}{\partial p_c} \tag{34}$$

$$\mathbf{C}_6 = 1 - \left(S_w + e\,\tfrac{\partial S_w}{\partial e}\right) \tag{35}$$

1.9.6 Load and flow vectors

$$\mathbf{F}_u = \int_{\Omega} \mathbf{N}_u^{\top} \rho\, \mathbf{b} \, d\Omega + \int_{\Gamma_t} \mathbf{N}_u^{\top} \bar{\mathbf{t}} \, d\Gamma \tag{36}$$

$$\mathbf{F}_w = \int_{\Omega} (\nabla \mathbf{N}_{pw})^{\top} \mathbf{k}_w \rho_w \mathbf{b} \, d\Omega - \int_{\Gamma_{qw}} \mathbf{N}_{pw}^{\top} \bar{\dot{\mathbf{w}}}^{w} \, d\Gamma \tag{37}$$

$$\mathbf{F}_g = \int_{\Omega} (\nabla \mathbf{N}_{pg})^{\top} \mathbf{k}_g \rho_g \mathbf{b} \, d\Omega - \int_{\Gamma_{qg}} \mathbf{N}_{pg}^{\top} \bar{\dot{\mathbf{w}}}^{g} \, d\Gamma \tag{38}$$

1.10 Table 2: Variable Definitions

Symbol	Description
$M^{\alpha},\ \Omega^{\alpha}$	mass, volume of phase $\alpha \in \{s, w, g\}$
$\rho_{\alpha},\ \rho^{\alpha},\ \rho$	phase density, partial density, mixture density
$n^{\alpha},\ n$	phase volume fraction, porosity
$S_w,\ S_g$	water and air degree of saturation
$p_w,\ p_g,\ p_c = p_g - p_w$	pore-water pressure, pore-air pressure, suction

Symbol	Description
$\boldsymbol{\sigma},\ \boldsymbol{\sigma}'$	total stress, effective stress
$\boldsymbol{\sigma}_{\text{net}}$	net stress $= \boldsymbol{\sigma} + p_g \mathbf{I}$
$\chi = \alpha_{\text{eff}} S_w$	effective-stress weighting factor
$\alpha_{\text{eff}}(S_w)$	saturation correction, $= S_w^{\frac{\beta_1}{S_w^{\beta_2}} - 1}$
$\beta_1,\ \beta_2$	material parameters in α_{eff}
$K_\beta,\ \eta_\beta$	bulk modulus and viscosity of phase β
$\mathbf{k}_{\text{int}},\ k_{r\beta},\ \mathbf{k}_\beta$	intrinsic, relative, and effective permeability
$\mathbf{u},\ \dot{\mathbf{u}},\ \ddot{\mathbf{u}}$	displacement, velocity, acceleration (solid)
$\mathbf{w}^\beta$	Darcy velocity of phase β
$\boldsymbol{\varepsilon}$	strain tensor
$\mathbf{b}$	body-force vector
$E_I,\ E_k,\ \mathcal{P}$	internal energy, kinetic energy, power
$\mathbf{N},\ \mathbf{B},\ \mathbf{L}$	shape, strain–displacement, differential operators
$\mathbf{M},\ \mathbf{K},\ \mathbf{C},\ \mathbf{H}$	global mass, stiffness, damping/coupling, conductivity matrices
$\mathbf{Q},\ \mathbf{R}$	solid–fluid coupling matrices
$\mathbf{M}_u,\ \mathbf{M}_w,\ \mathbf{M}_g$	phase-assembled mass matrices
$\mathbf{H}_{ww},\ \mathbf{H}_{gg}$	hydraulic conductivity matrices
$\mathbf{C}_{ww},\ \mathbf{C}_{gg},\ \mathbf{C}_{wg},\ \mathbf{C}_{gw}$	compressibility/coupling matrices
$\mathbf{C}_1^*,\ \mathbf{C}_2^*,\ \mathbf{C}_3^*,\ \mathbf{C}_6$	element-level coefficients (see Appendix A)
$\mathbf{n}^*,\ \mathbf{I}_\sigma^\top$	outward unit normal, stress-traction operator
$\bar{\mathbf{t}}$	prescribed traction (Neumann BC)
$\overline{\mathbf{w}}^\beta$	prescribed Darcy flux of phase β
$\mathbf{U},\ \mathbf{P}_w,\ \mathbf{P}_g$	global DOF vectors (displacement, p_w, p_g)
$\mathbf{F}_u,\ \mathbf{F}_w,\ \mathbf{F}_g$	global load / flow vectors
$\mathbf{K}_{NC_i},\ \mathbf{F}_{NC_i}$	contact stiffness and force (segment i)
e	void ratio
$\mathbf{m}$	Coupling vector $= [1,\ 1,\ 1,\ 0,\ 0,\ 0]^\top$ in 3D Voigt notation

1.11 References

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