



AD FALCON API Manual

Theory of Isotropic Linear Elasticity

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Contents

1	Theory of Isotropic Linear Elasticity	3
1.1	Syntax	3
1.2	Material parameters	3
1.3	Constitutive relation	3
1.3.1	Elastic moduli	3
1.3.2	Compliance matrix	4
1.3.3	Stiffness matrix	4
1.4	Stress–strain relation	5
1.4.1	Decomposition into volumetric and deviatoric parts	5
1.5	Parameter constraints	5
1.6	Plane strain and plane stress	6
1.6.1	Plane strain	6
1.6.2	Plane stress	6
1.7	Special cases	6
1.7.1	Incompressible material ($\nu = 0.5$)	6
1.7.2	Nearly incompressible material	6
1.7.3	Auxetic materials ($\nu < 0$)	7
1.8	References (selection)	7

1 Theory of Isotropic Linear Elasticity

This note describes the isotropic linear elastic constitutive model implemented in FALCON.

1.1 Syntax

This model is configured in % Materials using @UMAT: with category Mechanical (for example, LinearElasticUMAT).

```
@UMAT: /path/to/LinearElasticUMAT.cpp /path/to/LinearElasticUMAT.hpp
Mechanical \
  YoungsModulus=<E> PoissonsRatio=<nu>
```

To approximate a rigid solid, use a very large YoungsModulus and an appropriate PoissonsRatio.

See [Material Models: Syntax & Conventions](#) for shared rules (directive formatting, spacing, etc.).

1.2 Material parameters

Symbol	Keyword in input	Units	Required	Description
E	YoungsModulus	stress	✓	Young's modulus.
ν	PoissonsRatio	-	✓	Poisson's ratio.

1.3 Constitutive relation

For an **isotropic** linear elastic material, the mechanical properties are identical in all directions. The constitutive relation is expressed in Voigt notation (order: [11, 22, 33, 23, 13, 12]) as:

$$\sigma = C\varepsilon \quad (1)$$

where σ is the stress vector, ε is the strain vector, and C is the 6×6 stiffness matrix.

1.3.1 Elastic moduli

The elastic behavior is characterized by two independent material constants. The **bulk modulus** K and **shear modulus** G are related to Young's modulus E and Poisson's ratio ν by:

$$K = \frac{E}{3(1-2\nu)} \quad (2)$$

$$G = \frac{E}{2(1 + \nu)} \quad (3)$$

The bulk modulus represents the material's resistance to uniform compression, while the shear modulus represents its resistance to shear deformation.

1.3.2 Compliance matrix

The compliance matrix $S = C^{-1}$ relates strain to stress:

$$S = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1 + \nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1 + \nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1 + \nu) \end{bmatrix} \quad (4)$$

The shear components appear as $2(1 + \nu)$ due to the engineering shear strain convention in Voigt notation (where $\gamma_{ij} = 2\varepsilon_{ij}$ for $i \neq j$).

1.3.3 Stiffness matrix

The stiffness matrix C is the inverse of the compliance matrix. In terms of bulk modulus K and shear modulus G :

$$C = \begin{bmatrix} K + \frac{4}{3}G & K - \frac{2}{3}G & K - \frac{2}{3}G & 0 & 0 & 0 \\ K - \frac{2}{3}G & K + \frac{4}{3}G & K - \frac{2}{3}G & 0 & 0 & 0 \\ K - \frac{2}{3}G & K - \frac{2}{3}G & K + \frac{4}{3}G & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix} \quad (5)$$

Alternatively, the stiffness matrix can be expressed in terms of the **Lamé parameters** λ and μ :

$$C = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \quad (6)$$

where: - $\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$ (first Lamé parameter) - $\mu = G = \frac{E}{2(1+\nu)}$ (second Lamé parameter, equal to the shear modulus)

The relationship between the different representations is: - $K = \lambda + \frac{2}{3}\mu$ - $G = \mu$

1.4 Stress–strain relation

The incremental stress–strain relation is:

$$\Delta\sigma = C\Delta\varepsilon \quad (7)$$

where $\Delta\sigma$ is the stress increment and $\Delta\varepsilon$ is the strain increment.

1.4.1 Decomposition into volumetric and deviatoric parts

The stress and strain can be decomposed into volumetric (mean) and deviatoric parts:

Volumetric components:

$$\sigma_m = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}), \quad \varepsilon_v = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$$

Deviatoric components:

$$\mathbf{s} = \boldsymbol{\sigma} - \sigma_m \mathbf{I}, \quad \mathbf{e} = \boldsymbol{\varepsilon} - \frac{1}{3}\varepsilon_v \mathbf{I}$$

The constitutive relation separates into: - **Volumetric response:** $\sigma_m = K\varepsilon_v$ - **Deviatoric response:** $\mathbf{s} = 2G\mathbf{e}$

This separation is useful for understanding the material's response to different loading modes.

1.5 Parameter constraints

For the material to be physically admissible:

1. **Positive moduli:** Young's modulus must be positive:

$$E > 0$$

2. **Poisson's ratio bounds:** The Poisson's ratio must satisfy:

$$-1 < \nu < 0.5$$

Physical interpretation:

- $\nu > 0$: Material contracts laterally when stretched (typical for most materials)
- $\nu = 0$: No lateral contraction (e.g., cork)
- $\nu < 0$: Material expands laterally when stretched (auxetic materials)
- $\nu = 0.5$: Incompressible material (e.g., rubber)
- $\nu \geq 0.5$: Not physically admissible (would require infinite bulk modulus)

3. **Stability:** For thermodynamic stability, the bulk modulus and shear modulus must be positive:

$$K > 0 \Rightarrow \nu < 0.5$$

$$G > 0 \Rightarrow \nu > -1$$

1.6 Plane strain and plane stress

1.6.1 Plane strain

For plane strain conditions (e.g., $\varepsilon_{33} = 0$), the stress–strain relation reduces to a 3×3 system for $[\sigma_{11}, \sigma_{22}, \sigma_{12}]$ and $[\varepsilon_{11}, \varepsilon_{22}, \gamma_{12}]$:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} \quad (8)$$

The out-of-plane stress is:

$$\sigma_{33} = \nu(\sigma_{11} + \sigma_{22})$$

1.6.2 Plane stress

For plane stress conditions (e.g., $\sigma_{33} = 0$), the stress–strain relation becomes:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} \quad (9)$$

The out-of-plane strain is:

$$\varepsilon_{33} = -\frac{\nu}{1-\nu}(\varepsilon_{11} + \varepsilon_{22})$$

1.7 Special cases

1.7.1 Incompressible material ($\nu = 0.5$)

For an incompressible material, the bulk modulus becomes infinite ($K \rightarrow \infty$), and the volumetric strain is zero ($\varepsilon_v = 0$). The material response is purely deviatoric:

$$s = 2Ge, \quad \text{with } \varepsilon_v = 0$$

1.7.2 Nearly incompressible material

For materials with $\nu \approx 0.5$ (e.g., rubber, soft biological tissues), the bulk modulus is very large compared to the shear modulus. This can lead to numerical issues (volumetric locking) in

finite element simulations if not handled appropriately.

1.7.3 Auxetic materials ($\nu < 0$)

Auxetic materials have negative Poisson's ratios, meaning they expand laterally when stretched. While rare, such materials exist and are physically admissible as long as $\nu > -1$.

1.8 References (selection)

- Timoshenko, S. P. & Goodier, J. N. (1970). *Theory of Elasticity* (3rd ed.). McGraw-Hill.
- Gurtin, M. E. (1981). *An Introduction to Continuum Mechanics*. Academic Press.
- Bower, A. F. (2009). *Applied Mechanics of Solids*. CRC Press.

